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Propositional Logic as a Propositional Fuzzy Logic

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Abstract

There are several ways to extend the classical logical connectives for fuzzy truth degrees, in such a way that their behavior for the values 0 and 1 work exactly as in the classical one. For each extension of logical connectives the formulas which are always true (the tautologies) changes. In this paper we will provide a fuzzy interpretation for the usual connectives (conjunction, disjunction, negation, implication and bi-implication) such that the set of tautologies is exactly the set of classical tautologies. Thus, when we see logics as set of formulas, then the propositional (classical) logic has a fuzzy model.

Keywords: classical logic, fuzzy logic, weak t-norm.

1 Introduction

The fuzzy set theory introduced by Lofti Zadeh in [15] has as main characteristic the consideration of a degree of belief, i.e. a real value in $[0, 1]$, to indicate how much an expert believes that the element belongs to the set. This theory is appropriate to deal with concepts (and therefore with sets) not very precise such as the fat people, high temperatures, etc. In this way fuzzy logic, the subjacent logic, becomes an important tool to deal with uncertainty of knowledge and to represent the uncertainty of human reasoning.

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Two main directions can be distinguished in fuzzy logic [16]: 1) Fuzzy logic in the broad sense where the main goal is the development of computational systems based on fuzzy reasoning, such as fuzzy control systems and 2) Fuzzy logic in the narrow sense where fuzzy logic is seen as a symbolic logic and therefore questions as formal theories are studied. Lately, considerable progress has been made in strictly mathematical (formal and symbolic) aspects of fuzzy logic as logic with a comparative notion of truth [10].

Triangular norms (t-norms) were introduced by Schweizer and Sklar in [13] to model the distance in probabilistic metric spaces. But, Alsina, Trillas and Valverde in [1] showed that t-norms and their dual notion (t-conorm) can be used to model conjunction and disjunction in fuzzy logics generalizing several definitions for those connectives provided by Lotfi Zadeh in [15], Bellman and Zadeh in [4,5] and Yager in [14] (which define a general class of interpretations), etc. The other usual propositional connectives also can be fuzzy extended from a t-norm [8,12,6,3]. Thus, each t-norm determines a different set of true formulas (1-tautologies) and false formulas (0-contradictions) and therefore different (fuzzy) logics. The fuzzy logic where the interpretation of the propositional connectives are based on t-norm construction are known as triangular logics [9,2].

In this paper, we will consider the weak t-norm, and provide characterizations for the residuum, bi-implication, negation and t-conorm, all of them canonically obtained from this t-norm. Considering the usual propositional language, we will prove that interpreting the formulas based on these operators, each classical tautology is a tautology for this fuzzy interpretation. Since the converse is trivial, i.e. each 1-tautology (independently of the fuzzy extensions considered for the propositional connectives) is a tautology in the classical logic, we prove that the propositional classic logic (when understood as the set of tautologies) is a fuzzy logic, i.e. there exists a fuzzy interpretation for the propositional connectives such that the set of fuzzy tautologies coincides with the set of the classical tautologies.

2 Fuzzy logics

Let L_P be the usual propositional language. A **fuzzy evaluation** of propositional symbols PS is any function $e : PS \to \mathbb{I}$, where $\mathbb{I} = [0,1]$. Let $\mathcal{T} = \langle T, I, N, S, B \rangle$ be a fuzzy generalization of propositional connectives $\langle \wedge, \to, \neg, \vee, \leftrightarrow \rangle$, respectively. We can extend the evaluation e for a function $\mathcal{T}_e : L_P \longrightarrow \mathbb{I}$ as follows:

(i) $\mathcal{T}_e(p) = e(p)$ for each $p \in PS$,

(ii) $\mathcal{T}_e(\neg\alpha) = N(\mathcal{T}_e(\alpha))$,

(iii) $\mathcal{T}_e((\alpha \wedge \beta)) = T(\mathcal{T}_e(\alpha), \mathcal{T}_e(\beta))$,

(iv) $\mathcal{T}_e((\alpha \vee \beta)) = S(\mathcal{T}_e(\alpha), \mathcal{T}_e(\beta))$,

(v) $\mathcal{T}_e((\alpha \rightarrow \beta)) = I(\mathcal{T}_e(\alpha), \mathcal{T}_e(\beta))$, and

(vi) $\mathcal{T}_e((\alpha \leftrightarrow \beta)) = B(\mathcal{T}_e(\alpha), \mathcal{T}_e(\beta))$.

A formula $\alpha \in L_P$ is a 1-tautology w.r.t a $\mathcal{T}$, or simply $\mathcal{T}$-tautology, denoted by $\models_{\mathcal{T}} \alpha$, if for each fuzzy evaluation e, $\mathcal{T}_e(\alpha) = 1$. Thus, the fuzzy logic modelled by $\mathcal{T}$, or simply the $\mathcal{T}$-fuzzy logic is the set

$$LP_{\mathcal{T}} = \{\alpha \in L_P \ : \models_{\mathcal{T}} \alpha\}.$$

Proposition 2.1 *Let $\mathcal{T} = \langle T, I, N, S, B\rangle$ be a fuzzy generalization of propositional connectives and $\alpha \in L_P$. If $\models_{\mathcal{T}} \alpha$ then $\models \alpha$ (classical tautology).*

Proof. Straightforward. □

The propositional classical logic was defined in [7] as being the set of all tautologies. So, any fuzzy logic is contained in the classical one.

3 Equivalence between the propositional classical logic and the $\mathcal{W}$-fuzzy logic

Let $\mathcal{W} = \langle W, I_W, N_W, S_W, B_W\rangle$ be the fuzzy generalization of propositional connectives obtained canonically from the weak t-norm, i.e.

Conjunction:

$$W(x,y) = \begin{cases} \min\{x,y\} & \text{, if } \max\{x,y\} = 1 \\ 0 & \text{, otherwise} \end{cases}$$

Implication:

$$I_W(x,y) = \begin{cases} y & \text{, if } x = 1 \\ 1 & \text{, otherwise} \end{cases}$$

Negation:

$$N_W(x) = \begin{cases} 1 & \text{, if } x < 1 \\ 0 & \text{, if } x = 1 \end{cases}$$

Disjunction:

$$S_W(x,y) = \begin{cases} 1 \text{ , if } x = 1 \text{ or } y = 1 \\ 0 \text{ , otherwise} \end{cases}$$

Bi-implication:

$$B_W(x,y) = \begin{cases} y \text{ , if } x = 1 \\ x \text{ , if } y = 1 \\ 1 \text{ , otherwise} \end{cases}$$

Lemma 3.1 *Let* $\alpha, \beta, \gamma \in L_P$. *Then*

$A_1 \stackrel{\text{def}}{=} \alpha \to (\beta \to \alpha)$

$A_2 \stackrel{\text{def}}{=} (\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$

$A_3 \stackrel{\text{def}}{=} (\neg\beta \to \neg\alpha) \to ((\neg\beta \to \alpha) \to \beta)$

$A_4 \stackrel{\text{def}}{=} \alpha \wedge \beta \to \beta$

$A_5 \stackrel{\text{def}}{=} \alpha \to (\beta \to (\alpha \wedge \beta))$

$A_6 \stackrel{\text{def}}{=} \alpha \to (\alpha \vee \beta)$

$A_7 \stackrel{\text{def}}{=} \beta \to (\alpha \vee \beta)$

$A_8 \stackrel{\text{def}}{=} (\alpha \to \gamma) \to ((\beta \to \gamma) \to (\alpha \vee \beta \to \gamma))$

$A_9 \stackrel{\text{def}}{=} (\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha)$

$A_{10} \stackrel{\text{def}}{=} \neg\neg\alpha \to \alpha$

$A_{11} \stackrel{\text{def}}{=} (\alpha \leftrightarrow \beta) \to ((\alpha \to \beta) \wedge (\beta \to \alpha))$

$A_{12} \stackrel{\text{def}}{=} ((\alpha \to \beta) \wedge (\beta \to \alpha)) \to (\alpha \leftrightarrow \beta)$

are $\mathcal{W}$*-tautologies.*

Proof.

(i) Suppose that $\not\models_{\mathcal{W}} \alpha \to (\beta \to \alpha)$. Then, there is a fuzzy evaluation e such that $\mathcal{W}_e(\alpha \to (\beta \to \alpha)) \neq 1$. But, by definitions of I_W and $\mathcal{W}_e$, to it is necessary that $\mathcal{W}_e(\alpha) = 1$ and $\mathcal{W}_e(\beta \to \alpha) \neq 1$. But, by the same definitions, $\mathcal{W}_e(\beta \to \alpha) \neq 1$, only if $\mathcal{W}_e(\beta) = 1$ and $\mathcal{W}_e(\alpha) \neq 1$ leading to a contradiction. So, $\models_{\mathcal{W}} \alpha \to (\beta \to \alpha)$.

(ii) Suppose that $\not\models_{\mathcal{W}} (\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$. Then, $\mathcal{W}_e((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))) \neq 1$ for some fuzzy evaluation e. So, by definition of I_W and of $\mathcal{W}_e$, $\mathcal{W}_e(\alpha \to (\beta \to \gamma)) = 1$

and $\mathcal{W}_e((\alpha \to \beta) \to (\alpha \to \gamma)) \neq 1$. Being $\mathcal{W}_e(\alpha \to (\beta \to \gamma)) = 1$, necessarily $\mathcal{W}_e(\alpha) \neq 1$ and being $\mathcal{W}_e((\alpha \to \beta) \to (\alpha \to \gamma)) \neq 1$, then $\mathcal{W}_e(\alpha \to \beta) = 1$ and $\mathcal{W}_e(\alpha \to \gamma) \neq 1$. Thus, because $\mathcal{W}_e(\alpha \to \gamma) \neq 1$, $\mathcal{W}_e(\alpha) = 1$, also leading to a contradiction. So, $\models_{\mathcal{W}} (\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$.

(iii) Suppose that $\not\models_{\mathcal{W}} (\neg\beta \to \neg\alpha) \to ((\neg\beta \to \alpha) \to \beta)$. So for some fuzzy evaluation e, $\mathcal{W}_e((\neg\beta \to \neg\alpha) \to ((\neg\beta \to \alpha) \to \beta)) \neq 1$. Thus, by definition of I_W and of $\mathcal{W}_e$, $\mathcal{W}_e(\neg\beta \to \neg\alpha) = 1$ and $\mathcal{W}_e((\neg\beta \to \alpha) \to \beta) \neq 1$. But, by the same definitions, if $\mathcal{W}_e((\neg\beta \to \alpha) \to \beta) \neq 1$ then $\mathcal{W}_e(\neg\beta \to \alpha) = 1$ and $\mathcal{W}_e(\beta) \neq 1$. Because $\mathcal{W}_e(\neg\beta \to \alpha) = 1$, $\mathcal{W}_e(\neg\beta) = 1$ and $\mathcal{W}_e(\alpha) = 1$, or $\mathcal{W}_e(\neg\beta) \neq 1$. The last implies that $\mathcal{W}_e(\beta) = 1$, which is a contradiction. So, $\mathcal{W}_e(\neg\beta) = 1$ and $\mathcal{W}_e(\alpha) = 1$. On the other hand, since $\mathcal{W}_e(\neg\beta \to \neg\alpha) = 1$, or $\mathcal{W}_e(\neg\beta) = 1$ and $\mathcal{W}_e(\neg\alpha) = 1$. Therefore $\mathcal{W}_e(\alpha) \neq 1$ which is a contradiction, or $\mathcal{W}_e(\neg\beta) \neq 1$ which is also a contradiction. So, $\models_{\mathcal{W}} (\neg\beta \to \neg\alpha) \to ((\neg\beta \to \alpha) \to \beta)$.

(iv) Suppose that $\not\models_{\mathcal{W}} \alpha \wedge \beta \to \beta$. Then, by definition of I_W and of $\mathcal{W}_e$, for some fuzzy evaluation e, $\mathcal{W}_e(\alpha \wedge \beta) = 1$ and $\mathcal{W}_e(\beta) \neq 1$. Thus, by definition of $\mathcal{W}_e$, $W(\mathcal{W}_e(\alpha), \mathcal{W}_e(\beta)) = 1$ and therefore $_e(\alpha) = \mathcal{W}_e(\beta) = 1$ which is a contradiction. So, $\models_{\mathcal{W}} \alpha \wedge \beta \to \beta$.

(v) Suppose that $\not\models_{\mathcal{W}} \alpha \to (\beta \to (\alpha \wedge \beta))$. Then, by definition of I_W and of $\mathcal{W}_e$, for some fuzzy evaluation e, $\mathcal{W}_e(\alpha) = 1$ and $\mathcal{W}_e(\beta \to (\alpha \wedge \beta)) \neq 1$. But, because $\mathcal{W}_e(\beta \to (\alpha \wedge \beta)) \neq 1$, $\mathcal{W}_e(\beta) = 1$ and $\mathcal{W}_e(\alpha \wedge \beta) \neq 1$. So, because $\mathcal{W}_e(\alpha \wedge \beta) \neq 1$, $W(\mathcal{W}_e(\alpha), \mathcal{W}_e(\beta)) \neq 1$. Therefore, $\mathcal{W}_e(\alpha) \neq 1$ and $\mathcal{W}_e(\beta) \neq 1$ which is a contradiction. Hence, $\models_{\mathcal{W}} \alpha \to (\beta \to (\alpha \wedge \beta))$.

(vi) Suppose that $\not\models_{\mathcal{W}} \alpha \to (\alpha \vee \beta)$. Then, by definition of I_W and of $\mathcal{W}_e$, $\mathcal{W}_e(\alpha) = 1$ and $\mathcal{W}_e(\alpha \vee \beta) \neq 1$ for some fuzzy evaluation e. So, because $\mathcal{W}_e(\alpha \vee \beta) \neq 1$, $S_W(\mathcal{W}_e(\alpha), \mathcal{W}_e(\beta)) \neq 1$. Therefore $\mathcal{W}(\alpha) \neq 1$ and $\mathcal{W}(\beta) \neq 1$ which is a contradiction. Hence, $\models_{\mathcal{W}} \alpha \to (\alpha \vee \beta)$.

(vii) Analogously.

(viii) Suppose that $\not\models_{\mathcal{W}} (\alpha \to \gamma) \to ((\beta \to \gamma) \to (\alpha \vee \beta \to \gamma))$. Then, by definition of I_W and of $\mathcal{W}_e$, there exists a fuzzy evaluation e such that $\mathcal{W}_e(\alpha \to \gamma) = 1$ and $\mathcal{W}_e((\beta \to \gamma) \to (\alpha \vee \beta \to \gamma)) \neq 1$. Therefore, by definition of I_W, $\mathcal{W}_e(\beta \to \gamma) = 1$ and $\mathcal{W}_e(\alpha \vee \beta \to \gamma)) \neq 1$. So, by the same definition, $\mathcal{W}_e(\alpha \vee \beta) = 1$ and $\mathcal{W}_e(\gamma)) \neq 1$. By definition of N_W, $\mathcal{W}_e(\alpha) = 1$ or $\mathcal{W}_e(\beta) = 1$. If $\mathcal{W}_e(\alpha) = 1$, then because $\mathcal{W}_e(\alpha \to \gamma) = 1$ and by definition I_W, $\mathcal{W}_e(\alpha) \neq 1$ which is a contradiction, or $\mathcal{W}_e(\alpha) = \mathcal{W}_e(\gamma) = 1$ which also is a contradiction. So, $\mathcal{W}_e(\beta) = 1$. But, because, $\mathcal{W}_e(\beta \to \gamma) = 1$ and by definition of I_W, $\mathcal{W}_e(\beta) \neq 1$ which is a contradiction, or $\mathcal{W}_e(\beta) = \mathcal{W}_e(\gamma) = 1$ which also is a contradiction. So,

$\models_{\mathcal{W}} (\alpha \rightarrow \gamma) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \vee \beta \rightarrow \gamma))$.

(ix) Suppose that $\not\models_{\mathcal{W}} (\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \neg\beta) \rightarrow \neg\alpha)$. Then, by definition of I_W and of $\mathcal{W}_e$, for some fuzzy evaluation e, $\mathcal{W}_e(\alpha \rightarrow \beta) = 1$ and $\mathcal{W}_e((\alpha \rightarrow \neg\beta) \rightarrow \neg\alpha) \neq 1$. So, $\mathcal{W}_e(\alpha \rightarrow \neg\beta) = 1$ and $\mathcal{W}_e(\neg\alpha) \neq 1$. Thus, by definition of N_W, $\mathcal{W}_e(\alpha) = 1$ and, because $\mathcal{W}_e(\alpha \rightarrow \beta) = 1$, or $\mathcal{W}_e(\alpha) \neq 1$ which is a contradiction, or $\mathcal{W}_e(\alpha) = 1$ and $\mathcal{W}_e(\beta) = 1$. On the other hand, since $\mathcal{W}_e(\alpha \rightarrow \neg\beta) = 1$, or $\mathcal{W}_e(\alpha) \neq 1$ which is a contradiction or $\mathcal{W}_e(\alpha) = \mathcal{W}_e(\neg\beta) = 1$ and, by definition of N_W, $\mathcal{W}_e(\beta) \neq 1$ which also is a contradiction. Hence, $\models_{\mathcal{W}} (\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \neg\beta) \rightarrow \neg\alpha)$.

(x) Suppose that $\not\models_{\mathcal{W}} \neg\neg\alpha \rightarrow \alpha$. Then, by definition of I_W and of $\mathcal{W}_e$, for some fuzzy evaluation e, $\mathcal{W}_e(\neg\neg\alpha) = 1$ and $\mathcal{W}_e(\alpha) \neq 1$. But, because $\mathcal{W}_e(\neg\neg\alpha) = 1$, $\mathcal{W}_e(\neg\alpha) \neq 1$ and therefore $\mathcal{W}_e(\alpha) = 1$ which is a contradiction. So, $\models_{\mathcal{W}} \neg\neg\alpha \rightarrow \alpha$.

(xi) Suppose that $\not\models_{\mathcal{W}} (\alpha \leftrightarrow \beta) \rightarrow ((\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha))$. Then, by definition of I_W and of $\mathcal{W}_e$, for some fuzzy evaluation e, $\mathcal{W}_e(\alpha \leftrightarrow \beta) = 1$ and $\mathcal{W}_e((\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)) \neq 1$. Thus, by definition of B_W, $\mathcal{W}_e(\alpha) = \mathcal{W}_e(\beta) = 1$ or, $\mathcal{W}_e(\alpha) \neq 1$ and $\mathcal{W}_e(\beta) \neq 1$. On the other hand, by definition of weak t-norm, $\mathcal{W}_e(\alpha \rightarrow \beta) \neq 1$ or $\mathcal{W}_e(\beta \rightarrow \alpha) \neq 1$. So, by definition of I_W, or $\mathcal{W}_e(\alpha) = 1$ and $\mathcal{W}_e(\beta) \neq 1$ which is a contradiction, or $\mathcal{W}_e(\beta) = 1$ and $\mathcal{W}_e(\alpha) \neq 1$ which also is a contradiction. So, $\models_{\mathcal{W}} (\alpha \leftrightarrow \beta) \rightarrow ((\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha))$.

(xii) Suppose that $\not\models_{\mathcal{W}} (((\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)) \rightarrow (\alpha \leftrightarrow \beta))$. Then, by definition of I_W and of $\mathcal{W}_e$, $\mathcal{W}_e((\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)) = 1$ and $\mathcal{W}_e(\alpha \leftrightarrow \beta) \neq 1$, for some fuzzy evaluation e. Thus, by definition of B_W, or (*) $\mathcal{W}_e(\alpha) = 1$ and $\mathcal{W}_e(\beta) \neq 1$ or, (**) $\mathcal{W}_e(\alpha) \neq 1$ and $\mathcal{W}_e(\beta) = 1$. On the other hand, by definition of weak t-norm, $\mathcal{W}_e(\alpha \rightarrow \beta) = \mathcal{W}_e(\beta \rightarrow \alpha) = 1$. So, by definition of I_W, or $\mathcal{W}_e(\alpha) \neq 1$ or $\mathcal{W}_e(\alpha) = \mathcal{W}_e(\beta) = 1$, and, or $\mathcal{W}_e(\beta) \neq 1$ or $\mathcal{W}_e(\alpha) = \mathcal{W}_e(\beta) = 1$. So we have two cases: 1) $\mathcal{W}_e(\alpha) \neq 1$ and $\mathcal{W}_e(\beta) \neq 1$ which is contradiction with (*) as much as (**). 2) $\mathcal{W}_e(\alpha) = \mathcal{W}_e(\beta) = 1$, which also is a contradiction with (*) as much as (**). Therefore, $\models_{\mathcal{W}} (((\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)) \rightarrow (\alpha \leftrightarrow \beta))$.

□

Lemma 3.2 *Let $\alpha, \beta \in L_P$. If $\models_{\mathcal{W}} \alpha$ and $\models_{\mathcal{W}} \alpha \rightarrow \beta$ then $\models_{\mathcal{W}} \beta$.*

Proof. If $\models_{\mathcal{W}} \alpha$ and $\models_{\mathcal{W}} \alpha \rightarrow \beta$, then for each fuzzy evaluation e, $\mathcal{W}_e(\alpha) = 1$ and $\mathcal{W}_e(\alpha \rightarrow \beta) = 1$. But, if $\mathcal{W}_e(\alpha \rightarrow \beta) = 1$, then or $\mathcal{W}_e(\alpha) \neq 1$ which is a contradiction or $\mathcal{W}_e(\alpha) = 1$ and $\mathcal{W}_e(\beta) = 1$. So, $\mathcal{W}_e(\beta) = 1$. Therefore, $\models_{\mathcal{W}} \beta$. □

Observe that this lemma says that the modus ponens preserve $\mathcal{W}$-tautologies.

Theorem 3.3 *Let $\alpha \in L_P$. $\models \alpha$ if, only if, $\models_{\mathcal{W}} \alpha$.*

Proof. Consider the propositional formal theory (TP) describe by Kleene in [11], namely, $TP_K = \langle L_P, \Delta, MP\rangle$, where L_P is the propositional language, $\Delta = \{A_1, \ldots, A_{12}\}$ and MP is the modus ponens rule. As proved by Kleene, all tautology is a theorem of TP_K. If α is a theorem in TP_K then there exists a proof $\alpha_1, \ldots, \alpha_n$ of α in TP_K. We will prove by induction that for each $i = 1, \ldots, n$, $\models_{\mathcal{W}} \alpha_i$.

For $i = 1$, α_i is an axiom. So, by lemma 3.1, $\models_{\mathcal{W}} \alpha_1$.

Suppose that $\models_{\mathcal{W}} \alpha_i$ for each $i < k$. Then α_k or is an axiom, in whose case by lemma 3.1, $\models_{\mathcal{W}} \alpha_k$, or there exist $k_1, k_2 < k$ such that α_k is obtained in the proof as modus ponens of α_{k_1} and α_{k_2}. Therefore, $\alpha_{k_2} = \alpha_{k_1} \rightarrow \alpha_k$. By inductive hypothesis $\models_{\mathcal{W}} \alpha_{k_1}$ and $\models_{\mathcal{W}} \alpha_{k_2}$. So, by lemma 3.2, $\models_{\mathcal{W}} \alpha_k$.

Therefore, $\models_{\mathcal{W}} \alpha_i$ for each $i = 1, \ldots, n$. In particular $\models_{\mathcal{W}} \alpha_n$ (which is α). So, if α is a tautology then $\models_{\mathcal{W}} \alpha$. The reverse, i.e. if $\models_{\mathcal{W}} \alpha$ then α was proved in proposition 2.1. □

4 Final Remarks

The main contribution of this paper was to proved that the classical logic, when seen as the set of tautologies as in [7], can be also modelled by fuzzy connectives, and therefore is a fuzzy logic.

The importance of these results is to make possible to apply all the mathematical and computational tools developed for classical propositional logic (such as formal theories, automated theorem provers, programming logic languages, etc.) to the propositional fuzzy logics based on the weak t-norm (as seen in this paper). So, we can deal with (propositional) approximate reasoning as we can with the exact reasoning. In order to turn this work more expressive, in a further work, we will prove that the classical predicate logic can be seen (in the sense of this paper) as a fuzzy logic.

Apparently the main result of this paper is a trivial consequence of identify 1 with 1 and the other values with 0, making the behavior of the fuzzy connectives to coincide with the classical one, or more formally, because given the function $k : \mathbb{I} \rightarrow \{0, 1\}$ defined by $k(1) = 1$ and $k(x) = 0$ for each $x \in [0, 1)$, the following equation is satisfied for each formula $\alpha \in L_P$ and evaluation e:

$$k \circ \mathcal{W}_e(\alpha) = C_{k \circ e}(\alpha) \tag{1}$$

where C_f is the classical extension of a classical evaluation f. Nevertheless, this equation is also satisfied for a natural fuzzy extension based on the product t-norm and for $k : \mathbb{I} \rightarrow \{0, 1\}$ defined by $k(0) = 0$ and $k(x) = 1$ for each

$x \in (0, 1]$. But, the classical tautology $\neg\neg\alpha \rightarrow \alpha$ is not a tautology for this fuzzy logic.

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